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IIT JEE | MEDICAL | FOUNDATION

JEE Advanced 2020

Maths Paper - 2_Solutions

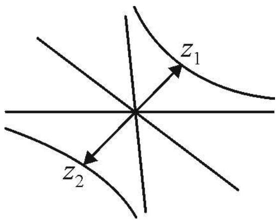
1. $\left[z^2+|z|^2\right]\left[z^2-|z|^2\right]=4iz^2$

Let $z=x+iy$

$\Rightarrow \quad \left[x^2-y^2+2ixy+x^2+y^2\right]\left[x^2-y^2+2ixy-x^2-y^2\right]=4iz^2$

$\Rightarrow \quad \left[2x^2+2ixy\right]\left[2ixy-2y^2\right]=4iz^2 \Rightarrow \quad 2x[x+iy].2yi[x+iy]=4iz^2$

$\Rightarrow \quad 4ixy.z^2=4iz^2 \therefore \quad xy=1$



$L_{TA}=|z_1-z_2|_{\min}=2\sqrt{2}$

$\therefore \quad |z_1-z_2|^2{}_{\min}=8$

2. Let "n" be the number of missiles required

$P(\text{hitting the tangent})=3/4$

$P(\text{atleast three succesful hits})\geq\frac{95}{100}$

$1-P(x=0)-P(x=1)-P(x=2)\geq\frac{95}{100}$

$^nc_0\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^n+{}^nc_1\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{n-1}+{}^nc_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2}\leq\frac{1}{20}$

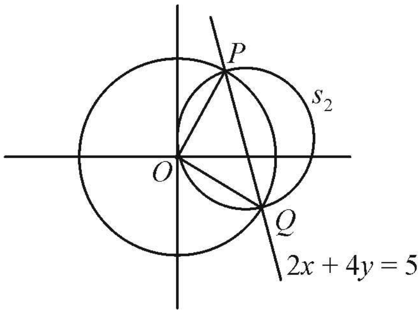
$\left(\frac{1}{4}\right)^n+n.\frac{3}{4}.\left(\frac{1}{4}\right)^{n-1}+\frac{n(n-1)}{2}.\frac{9}{16}\left(\frac{1}{4}\right)^{n-2}\leq\frac{1}{20}$

$\frac{1+3n+\frac{9n(n-1)}{2}}{4^n}\leq\frac{1}{20}$

$4^{n-1}\geq 5+15n+\frac{45n(n-1)}{2}$

$n_{least}=6$

3. Equation of $s_2 : \left(x^2+y^2-r^2\right)+\lambda\left(2x+4y-5\right)=0$



$(0,0): -r^2-5\lambda=0 \Rightarrow \lambda=-r^2/5$

Now $s_2=\left(x^2+y^2-r^2\right)-\frac{r^2}{5}\left(2x+4y-5\right)=0$

Centre : $\left[\frac{r^2}{5},\frac{2r^2}{5}\right]$ lies on

$x+2y-4 \Rightarrow \frac{r^2}{5}+\frac{4r^2}{5}=4 \Rightarrow r^2=4$

$\Rightarrow \quad \therefore \quad r=2$

4. $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Given : $a+d=3$

$A^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

$A^3=\begin{bmatrix} a^2+bc & 3b \\ 3c & bc+d^2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^3=\begin{bmatrix} a^3+abc+3bc & a^2b+b^2c+3bd \\ 3ac+bc^2+cd^2 & 3bc+bcd+d^3 \end{bmatrix}$

$tr.\left(A^3\right)=a^3+abc+3bc+3bc+bcd+d^3$

$=a^3+bc\left[a+b+d\right]+d^3$

$tr\left(A^3\right)=a^3+9bc+d^3=-18$

$=a^3+9bc+\left(3-a\right)^3=-18$

$=a^3+9bc+27-a^3-27a+9a^2=-18$

$=9bc-27a+9a^2=-45$

$=bc-3a+a^2=-5$

Now $|A|=ad-bc$

$=a\left(3-a\right)-bc$

$=3a-a^2-bc$

$=5$

5. $f(x)\begin{cases} -4x & ; \quad -1 < x \le -\dfrac{1}{2} \\ 2 & ; \quad -\dfrac{1}{2} < x < \dfrac{1}{2} \\ 4x & ; \quad \dfrac{1}{2} \le x < 1 \end{cases}$

$$g(x) = \begin{cases} x+1 & ; \quad -1 < x < 0 \\ x & ; \quad 0 \le x < 1 \end{cases}$$

$$f(g(x)) = \begin{cases} -4g(x) & ; \quad -1 < g(x) \le \dfrac{-1}{2} \\ 2 & ; \quad \dfrac{-1}{2} < g(x) < \dfrac{1}{2} \\ 4g(x) & ; \quad \dfrac{1}{2} \le g(x) < 1 \end{cases}$$

$$f(g(x)) = \begin{cases} 2 & ; \quad -1 < x < \dfrac{-1}{2} \\ 4(x+1) & ; \quad -\dfrac{1}{2} \le x < 0 \\ 2 & ; \quad 0 \le x < \dfrac{1}{2} \\ 4x & ; \quad \dfrac{1}{2} \le x < 1 \end{cases}$$

Discontinuous at $x = 0$ only $\therefore \ c = 1$

Non differentiable at $x = \dfrac{-1}{2}, 0, \dfrac{1}{2} \therefore d = 3$

$\therefore \qquad c + d = 4$

6. $$\lim_{x \to \frac{\pi}{2}} \frac{4\sqrt{2}.2\sin 2x.\cos x}{\left[\cos\dfrac{x}{2} - \cos\dfrac{7x}{2} + \cos\dfrac{5x}{2} - \cos\dfrac{3x}{2}\right] - \sqrt{2}\left[1 + \cos 2x\right]}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{8\sqrt{2}\sin 2x.\cos x}{2\cos\dfrac{3x}{2}\cos x - 2\cos\dfrac{5x}{2}\cos x - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{4\sqrt{2}\sin 2x}{\cos\dfrac{3x}{2} - \cos\dfrac{5x}{2} - \sqrt{2}\cos x}$$

$$\lim_{x \to \frac{\pi}{2}} \frac{8\sqrt{2}\cos 2x}{\dfrac{-3}{2}\sin\dfrac{3x}{2} + \dfrac{5}{2}\sin\dfrac{5x}{2} + \sqrt{2}\sin x}$$

$$\frac{-8\sqrt{2}}{\dfrac{-3}{2}.\dfrac{1}{\sqrt{2}} - \dfrac{5}{2}.\dfrac{1}{\sqrt{2}} + \sqrt{2}} = 8$$

7. $f'(x) = \dfrac{f(x)}{b^2 + x^2}$

$$\int_0^x \frac{f'(x)}{f(x)}dx = \int_0^x \frac{1}{b^2+x^2}dx$$

$$\ell n\,|f(x)| = \frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)$$

$f(x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$ as $f(0)=1$

Obviously $f'(x) > 0$ and

$f(x)f(-x) = 1$

8. $e^2 = 1 + \frac{b^2}{a^2}$ and $0 < b < a$

$\Rightarrow \quad 1 < e^2 < 2 \Rightarrow 1 < e < \sqrt{2}$

Slope of tangent at $P(x_1, y_1)$ is

$\frac{b^2}{a^2}\frac{x_1}{y_1}$ and that of normal is $-\frac{a^2 y_1}{b^2 x_1}$

Given $a^2 y_1 = b^2 x_1$

$x_1 = \frac{a^2}{\sqrt{a^2-b^2}}$ and $y_1 = \frac{b^2}{\sqrt{a^2-b^2}}$

Equation of tangent is $x - y = \sqrt{a^2-b^2}$

$\Rightarrow \quad a^2 - b^2 = 1$

Equation of normal is $x + y = \frac{a^2+b^2}{\sqrt{a^2-b^2}}$

$\Rightarrow \quad x + y = a^2 + b^2$

Notice the given triangle is an isosceles right triangle

$$\Delta = \frac{1}{2}\left(\frac{2b^2}{\sqrt{2}}\right)^2 = b^4$$

9. Given $f(x+y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = x\,g(x)$

So, $\lim_{x\to 0} f(x) = 0$ and $f(0) = 0\,g(0) = 0$

So, f is continuous at 0.

$$f'(0) = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{h\,g(h)}{h} = \lim_{h\to 0}\left(g(h)\right) = 1$$

So, $f'(0)$ exists and equals 1

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h) + f(x)f(h)}{h}$$

$= (1 + f(x)) \lim_{h \to 0} \frac{f(h)}{h}$

$f'(x) = 1 + f(x)$ so f is differentiable

$\Rightarrow \quad f(x) = e^x - 1 \quad \Rightarrow \quad f'(1) = e$

$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\frac{f(h)}{h} - 1}{h}$

$= \lim_{h \to 0} \frac{e^h - 1 - h}{h^2} = \frac{1}{2}$. So g is differentiable at zero.

It's almost obvious that g is differentiable elsewhere. A, B, D

10. Line joining $(3, 2, -1)$ to $(1, 0, -1)$ is normal to the plane

$\frac{\alpha}{2} = \frac{\beta}{2} \text{and} \, \gamma = 0$

$\Rightarrow \quad \alpha = \beta \quad \Rightarrow \quad \alpha = 1 = \beta$

Also mid-point of $(3, 2, -1)$ and $(1, 0, -1)$ lies on $\alpha x + \beta y + \gamma z = \delta$

$\Rightarrow \quad \delta = 2\alpha + \beta = 3$

11. $|\overrightarrow{PQ}| = |\overrightarrow{PS}| = \sqrt{a^2 + b^2}$

Since adjacent sides are equal

So, it's a rhombus.

Area = $\frac{1}{2} |\vec{D}_1 \times \vec{D}_2| = 2ab$

$\Rightarrow \quad ab = 4$

$\Rightarrow \quad |\vec{u}| = \frac{a + b}{\sqrt{a^2 + b^2}}$ and $|\vec{v}| = \left| \frac{a - b}{\sqrt{a^2 + b^2}} \right|$

$|\vec{\omega}| = \sqrt{2} \Rightarrow |\vec{u}| + |\vec{v}| = |\vec{\omega}|$

$\Rightarrow \quad (a + b) + |a - b| = \sqrt{2(a^2 + b^2)}$

$\Rightarrow \quad 2 \max\{a, b\}^2 = \sqrt{2(a^2 + b^2)}$

$(\max\{a, b\})^2 = (\min\{a, b\})^2$

$\Rightarrow \quad a = b = 2$

$\overrightarrow{PR} = 2a\hat{i} = 4\hat{i} \Rightarrow |\overrightarrow{PR}| = 4$

Angle bisector of $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is along $\hat{i}$ and hence D is wrong.

12. $f(m, n, p) = \sum_{i=0}^{p} \binom{m}{i} \binom{n + i}{p} \binom{p + n}{p - i}$

$$\sum_{i=0}^{p}\frac{m}{i!(m-i)!}\frac{(n+i)!}{p!(n+i-p)!}\frac{(p+n)!}{(p-i)!(n+i)!}$$

$$={}^{p+n}C_p\sum_{i=0}^{p}{}^{m}C_i\ {}^{n}C_{p-i}$$

$$={}^{p+n}C_p.{}^{m+n}C_p$$

$$f(n,m,p)={}^{p+m}C_p\ {}^{m+n}C_p$$

$$g(m,n)=\sum_{p=0}^{m+n}{}^{m+n}C_p=g(n,m)$$

$$=2^{m+n}$$

13. $1,2,3,4,5,6,7,.........,15$

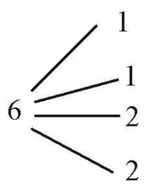
We have to select 4 numbers such that no two of them are consecutive

$$\begin{array}{ccccc} x_1 + & x_2 + & x_3 + & x_4 + & x_5 \;=11 \\ \ge 0 & \ge 1 & \ge 1 & \ge 1 & \ge 0 \end{array}$$

$${}^{11-3+4}C_4={}^{12}C_4=\frac{12\times 11\times 10\times 9}{24}$$

$$=495$$

14.



$$\frac{6!}{1!1!2!2!2!\times 2!}\ 4!=1080$$

15. $p(prime)=\dfrac{15}{36}$, $p(\text{square})=\dfrac{7}{36}$

$$p\left(\text{odd square/square before prime}\right)\quad \frac{=\dfrac{4}{36}+\dfrac{14}{36}\times\dfrac{4}{36}+\dfrac{4}{36}\left(\dfrac{14}{36}\right)^2.....}{\dfrac{7}{36}+\dfrac{14}{36}\times\dfrac{7}{36}+\left(\dfrac{14}{36}\right)^2\times\dfrac{7}{36}......}$$

$$=\frac{4}{7}$$

16. $f(x)=\dfrac{4^x}{4^x+2}$

$$f(x)+f(1-x)=1$$

$$f\left(\frac{1}{40}\right)+f\left(\frac{39}{40}\right)=1$$

$$19(1)+f\left(\frac{20}{40}\right)-f\left(\frac{1}{2}\right)=19$$

17. $$\int_0^{\pi} \left(f'(x) + F(x) \right) \cos x \, dx = 2$$

$$\boxed{\begin{array}{l} F(x) = \int_0^{x} f(x)\, dx \\ F'(x) = f(x) \end{array}}$$

$$\int_0^{\pi} \underset{\text{II}}{f'(x)} \ \underset{\text{I}}{\cos x} \ dx + \int_0^{\pi} F(x) \cos x \ dx = 2$$

$$\left[\cos x \ f(x) \right]_0^{\pi} + \int_0^{\pi} \underset{\text{I}}{\sin x} \ \underset{\text{II}}{f(x)} \ dx + \int_0^{\pi} F(x) \cos x \ dx = 2$$

$$-1 f(\pi) - f(0) + \left[\sin x \ F(x) \right]_0^{\pi} - \int_0^{\pi} \cos x \ F(x)\, dx + \int_0^{\pi} \cos x \ F(x)\, dx = 2$$

$$6 - f(0) = 2$$

$$f(0) = 4$$

18. $$f(\theta) = 1 + \sin 2\theta + \left(1 - \sin 2\theta\right)^2 \ \theta \in \left(0, \pi\right)$$

$$= 2 + \sin^2 2\theta - \sin 2\theta$$

$$f'(\theta) = 4 \sin 2\theta \cos 2\theta - 2 \cos 2\theta$$

$$= 2 \cos 2\theta \left(2 \sin 2\theta - 1 \right)$$

0		$\frac{\pi}{12}$		$\frac{\pi}{4}$		$\frac{5\pi}{12}$		$\frac{3\pi}{4}$		π
	$-$		$+$		$-$		$+$		$-$	

Local minimum at $\frac{\pi}{12}, \frac{5\pi}{12}$

$$\lambda_1 + \lambda_2 = \frac{6}{12} = 0.5$$