SOLUTIONS

Joint Entrance Exam | IITJEE-2023

25th JAN 2023 | Morning Shift

PHYSICS

SECTION - 1

1.(2) $\lambda = \dfrac{h}{\sqrt{2meV}}$

$\lambda_0 = \dfrac{h}{\sqrt{2me(20)}}, \ \lambda' = \dfrac{h}{\sqrt{2me(40)}} \qquad \therefore \dfrac{\lambda_0}{\lambda'} = \sqrt{\dfrac{40}{20}} \Rightarrow \lambda' = \dfrac{\lambda_0}{\sqrt{2}}$

2.(3) Photodiodes are used in reverse bias for meaning light intensity.

3.(4) $\vec{g}$ inside earth $= \dfrac{-GMx}{\text{Re}^3}$

Comparing with $a = -\omega^2 x$

$\omega = \sqrt{\dfrac{GMe}{Re^3}}, \ \dfrac{GMe}{R^2} = g$

$\omega = \sqrt{\dfrac{g}{\text{Re}}} \Rightarrow T = \dfrac{2\pi}{\omega}$

$T = 2\pi\sqrt{\dfrac{\text{Re}}{g}} \Rightarrow 2\pi\sqrt{\dfrac{6400\times10^3}{10}}$

$T = 2\pi\times800\text{sec} \qquad \therefore T = \dfrac{2\pi\times800}{3600}hr = 1.4hr$

4.(1) $n\lambda = \dfrac{yd}{D} \qquad \therefore 5\lambda = \dfrac{(5\times10^{-2})d}{1}$

$d = \dfrac{5\times600\times10^{-9}}{5\times10^{-2}}$

$d = 60\mu m$

5.(3) $\dfrac{d\theta}{dt} = -k(\theta-\theta_0)$

$\theta_0 \rightarrow$ temperature of surrounding

$\ln\left(\dfrac{\theta-\theta_0}{\theta_i-\theta_0}\right) = -kt$

$\theta = \theta_0 + (\theta_i - \theta_0)e^{-kt}$

$84 = 22 + (96-22)e^{-k\times2}$

$e^{-2k} = \dfrac{62}{74} \qquad \ldots\text{(i)}$

$68 = 22 + (74-22)e^{-kt}$

$e^{-kt} = \dfrac{46}{52} \qquad \ldots\text{(ii)}$

$-2 = \ln\dfrac{62}{74}$

$-kt = \ln\dfrac{46}{52}$

$\therefore t = 1.4\text{min}$

6.(2) $V = IR$

$R = \frac{\rho l}{A}$

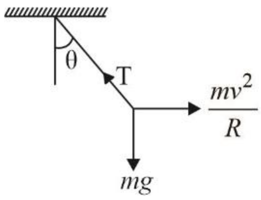
$d = \frac{M}{vol} = \frac{M}{A \times l} \Rightarrow A = \frac{M}{dl}$

$R = \frac{\rho l}{M} dl = \frac{\rho d l^2}{M}$

$3.4 = \frac{2 \times 1.7 \times 10^{-8} \times 8.92 \times 10^3 l^2}{8.92 \times 10^{-3}}$

$l = 10m$

7.(1) $T\cos\theta = mg \quad T\sin\theta = \frac{mv^2}{R}$



$\tan\theta = \frac{V^2}{Rg} = \frac{(20)^2}{40 \times 10} = 1$

$\theta = \frac{\pi}{4}$

8.(3) $t_1 = \frac{x}{V_1} \quad t_2 = \frac{x}{V_2}$

Average speed $= \frac{x + x}{t_1 + t_2} = \frac{2x}{\frac{x}{V_1} + \frac{x}{V_2}} = \frac{2V_1V_2}{V_1 + V_2}$

9.(4) $B = \mu_0 ni$

Magnetic intensity $= \frac{B}{\mu_0} = ni = \frac{1200}{2} \times 2 = 1.2 \times 10^3 Am^{-1}$

10.(4) $A \rightarrow \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I}{2r} = \frac{\mu_0 I}{2\pi r}[\pi - 1]$

$B \rightarrow \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{4r} = \frac{\mu_0 I}{4\pi r}(\pi + 2)$

$C \rightarrow \frac{\mu_0 I}{4\pi r}(\pi + 1)$

$D \rightarrow \frac{\mu_0 I}{4r}$

11.(2) Bandwidth $= 2f_m = 2 \times 5KHz = 10KHz$

12.(3) $\omega_0 = \frac{1}{\sqrt{L_0C_0}} \qquad \omega = \frac{1}{\sqrt{LC}}$

$\omega = \frac{1}{\sqrt{2L_0 \times 8C_0}} = \frac{1}{4\sqrt{L_0C_0}}$

$\omega = \frac{\omega_0}{4}$

13.(2) $V_{rms} = \sqrt{\frac{3RT}{M}}$

14.(3) $S = \frac{F}{l} = \frac{kg\,m s^{-2}}{m}$

$S = kg\,s^{-2}$

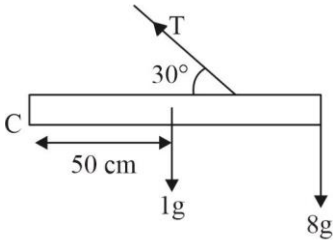
$P = \frac{F}{A} = kg\,m^{-1}s^{-2}$

$F = \eta A\frac{dV}{dx} \Rightarrow \eta = \frac{kg\,m s^{-2}}{m^2}$

$\eta = kg\,m^{-1}s^{-1}$

$I = mV = kg\,m s^{-1}$

15.(3) $\tau_C = 0$



$(T\sin 30°)\times 60 - 2g\times 50 + 8g\times 100$

$T = \frac{9000}{30} = 300\,N$

16.(1) Density of nucleus is same for all elements.

17.(2) $\eta = \frac{T_H - T_L}{T_H}$

$0.5 = \frac{600 - T_L}{600}$

$T_L = 300\,K$ [Sink]

$0.7 = \frac{T - 300}{T}$ $\quad\therefore\ T = 1000\,K$

18.(1) $\hat{C}\times\hat{E} = \hat{B}$

19.(4) $C = \frac{K\varepsilon_0 A}{d} + \frac{\varepsilon_0 A}{d}$

$d = 1mm,\ k = 5,\ A = 40cm^2$

$C = \frac{5\varepsilon_0\times 40\times 10^{-4}}{10^{-3}} + \frac{\varepsilon_0\times 40\times 10^{-4}}{10^{-3}}$

$C = 20\varepsilon_0 + 4\varepsilon_0$

$C = 24\varepsilon_0 F$

20.(3) $T = 2\pi\sqrt{\frac{l}{g}}$

At surface $g = \frac{GM}{R^2}$

$T = 2\pi\sqrt{\frac{lR^2}{GM}}$

At height R

$T' = 2\pi\sqrt{\frac{l(2R)^2}{GM}} = 2T$

SECTION – 2

21.(52)

22.(5) $y = \frac{Fl}{A\Delta l}$

$\Delta l = \frac{l}{Ay}F$

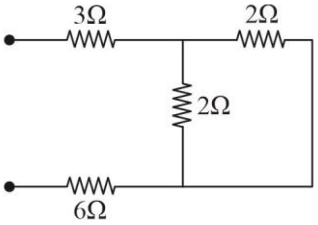
$\frac{l}{Ay} = 1$

$\frac{0.628}{\pi\frac{(4\times10^{-3})^2}{4}\times 4} = 1$

$y = \frac{0.628\times10^6}{\pi\times 4}$

$y = 5\times10^4\, N/m^2$

23.(10)



$R_{AB} = 10\Omega$

24.(2) $d\omega = (F\cos\theta)dx$

$\omega = \Delta KE$

$\omega = \int F(\cos kx)dx$

$\omega = 2\int \cos kx\, dx$

$\omega = \frac{2}{k}\sin kx$

$n = 2$

25.(45) $\frac{1}{2}mv^2 = 0.5eV$

$V_x = \sqrt{\frac{1eV}{m}}$

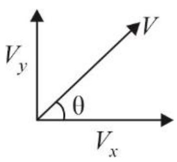
Time taken $= \frac{l}{V_x} = \frac{0.1}{V_x}$

Speed in vertical direction

$V_y = u + at$

$a = \frac{eE}{m} = \frac{10e}{m}$

$V_y = \frac{10e}{m}\times\frac{0.1}{V_x}$



$\tan\theta = \frac{V_y}{V_x}$

$\tan\theta = \frac{10e}{m}\times\frac{0.1}{V_x^2} = 1$

$\theta = 45°$

26.(100) $I = \dfrac{E}{Z}$

$$Z = \sqrt{(X_L - X_C)^2 + R^2}$$

For I_{max}, $Z \to \min$ $\quad \therefore X_L = X_C \Rightarrow \omega L = \dfrac{1}{\omega C}$

$$2\pi \times 2 \times 10^3 \times L = \frac{1}{2\pi \times 2 \times 10^3 \times 62.5 \times 10^{-9}}$$

$$L = \frac{10^8}{4\pi^2 \times 4 \times 10^6} = \frac{1000}{16 \times 10 \times 62.5}$$

$$L = \frac{1}{10} H = 100\, mH$$

27.(18) $\dfrac{2\pi}{\lambda}\Delta x = \dfrac{\pi}{3}$

$$\lambda = 6 \times \Delta x = 6 \times 6 = 36m$$

$$V = f\lambda = 36 \times 500 = 18 km/s$$

28.(4) $\vec{P} \times \vec{Q} = \begin{vmatrix} i & j & k \\ 3 & \sqrt{3} & 2 \\ 4 & \sqrt{3} & 2.5 \end{vmatrix} = 0.5\sqrt{3}\hat{i} + 0.5\hat{j} - \sqrt{3}\hat{k}$

Unit vector $\dfrac{0.5}{\sqrt{16}}\left(\sqrt{3}\hat{i} + \hat{j} - 2\sqrt{3}\hat{k}\right) = \dfrac{1}{8}\left(\sqrt{3}\hat{i} + \hat{j} - 2\sqrt{3}\hat{k}\right) = 8$

29.(27) $\dfrac{1}{\lambda_0} = R\left(\dfrac{1}{4} - \dfrac{1}{9}\right)$

$$\frac{1}{\lambda_0} = \frac{5R}{36}$$

$$\frac{1}{\lambda'} = R\left(\frac{1}{9} - \frac{1}{16}\right)$$

$$\frac{1}{\lambda'} = R\left(\frac{7}{9 \times 16}\right)$$

$$\frac{\lambda'}{\lambda_0} = \frac{5 \times 9 \times 16}{36 \times 7}$$

$$\frac{\lambda'}{\lambda_0} = \frac{20}{7}$$

$$x = 7$$

30.(17) $I_{AB} = \dfrac{MR^2}{2} + M\left(\dfrac{2R}{3}\right)^2$

$$I_{AB} = \frac{MR^2}{2} + \frac{4MR^2}{9}$$

$$I_{AB} = \frac{17MR^2}{18} \quad \therefore x = 17$$

CHEMISTRY

SECTION - 1

1.(2) Nucleophilic aromatic substitution takes place if an electron withdrawing group (EWG) like $-NO_2$ is present at ortho/para position. Hence in option (2) the rate will be minimum.

2.(1) In aqueous medium basic strength of methyl amines follow the below order

$2° > 1° > 3° > NH_3$

3.(4) Flame Test

$K \rightarrow$ Violet colour

$Ca \rightarrow$ Brick red

$Sr \rightarrow$ Crimson red

$Ba \rightarrow$ Apple green

4.(2) In order to remove silica during the metallurgy of copper below reaction occurs

$FeO + SiO_2 \longrightarrow FeSiO_3$

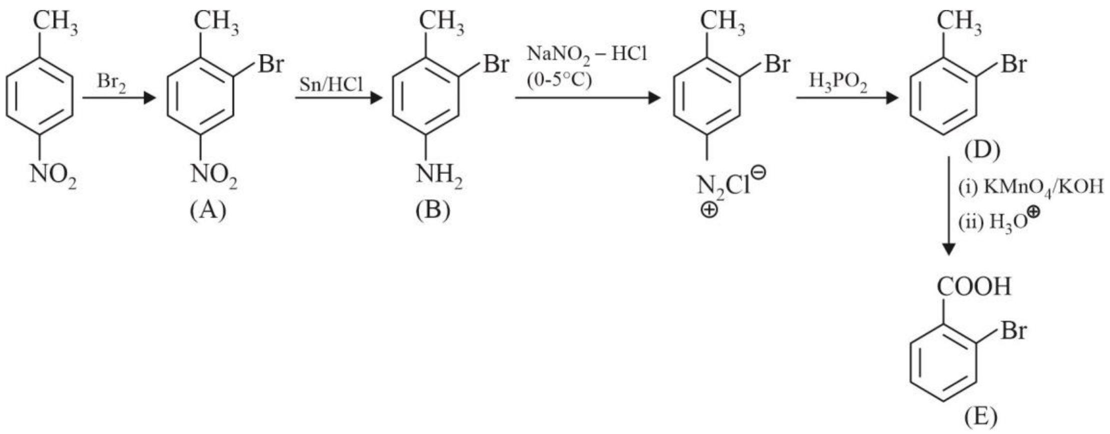
5.(1) Volume strength = Molarity $\times 11.2$

$$25 = \frac{n_{solute}}{V_{solution}} \times 11.2$$

$$n_{solute} = \frac{25}{11.2}$$

$$g_{solute} = \frac{25}{11.2} \times 34 = 75.89 \cong 75$$

6.(2)



7.(4) As the concentration of the substrate increases, the rate of enzyme catalyzed reaction also increases upto a certain point.

8.(2) X at alternate corners $= 4 \times \frac{1}{8} = \frac{1}{2}$

X at body center = 1

Y at $\frac{1}{3}$rd faces $= 6 \times \frac{1}{3} \times \frac{1}{2} = 1$

$X_{1.5}Y_1$

X_3Y_2

9.(2) $A : \alpha - D$ glucopyranose

$B : \beta - D$ glucopyranose

$C : \beta - D$ fructofuranose

$D : \alpha - D$ fructofuranose

10.(3)

	Group	Group reagent
Pb^{2+} ; Cu^{2+}	II	H_2S / HCl
Al^{3+} ; Fe^{3+}	III	NH_4OH / NH_4Cl
Co^{2+} ; Ni^{2+}	IV	H_2S / NH_4OH
Ba^{2+} ; Ca^{2+}	V	$(NH_4)_2CO_3$ / NH_4OH

11.(3) Staggered anti conformer is most stable in all $C_2 - C_3$ conformers of butane.

12.(1) $NO_2 \xrightarrow{\text{Sunlight}} \overset{(Y)}{NO} + \overset{(X)}{O} \xrightarrow{O_2(g)(A)} O_3(g)(B)$

13.(3) $r \propto \dfrac{n^2}{z}$

$r_{Li^{2+}} \propto \dfrac{2^2}{3} \qquad r_{Be^{3+}} \propto \dfrac{3^2}{4}$

$\dfrac{r_{Be^{3+}}}{r_{Li^{2+}}} = \dfrac{3^2}{4} \times \dfrac{3}{2^2}$

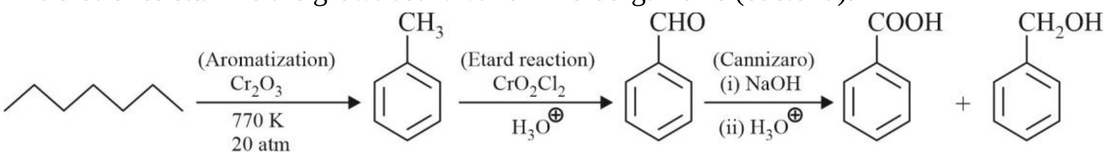
$r_{Be^{3+}} = \dfrac{27}{16} x$

14.(2) Order of e^- gain enthalpy

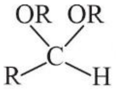
$Ne > Kr > Xe > He$

15.(2) Antibiotic resists/kills the growth/survival of microorganisms (bacteria).

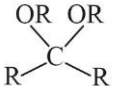
16.(2)



17.(4)



(acetal)



(ketal)

These are stable in basic medium as leaving group tendency of RO^- (alkoxide ion) is less.

18.(3) $\underset{(A)}{Ca(OH)_2} + 2NH_4Cl \longrightarrow CaCl_2 + 2H_2O + \underset{(B)}{2NH_3}$

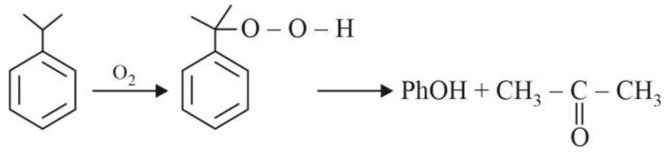
$\underset{(B)}{NH_3} + CO_2 + H_2O \longrightarrow \underset{(C)}{(NH_4)HCO_3}$

$\underset{(C)}{(NH_4)HCO_3} + NaCl \longrightarrow NH_4Cl + NaHCO_3$

19.(1) $SOCl_2 + P_4 \longrightarrow \underset{(A)}{PCl_3} + SO_2 + S_2Cl_2$

$\underset{(A)}{PCl_3} + H_2O \longrightarrow \underset{(B)}{H_3PO_3} + HCl$

20.(4)



SECTION – 2

21.(42) Percentage of sulphur $= \dfrac{32 \times x}{233 \times m} \times 100\%$

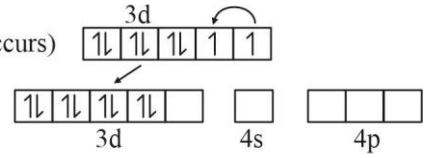
$x \rightarrow$ mass of $BaSO_4$ formed, m = Mass of organic compound

% of sulphur $= \dfrac{32 \times 1.4439}{233 \times 0.471} \times 100\% = 42.10\%$

$\simeq 42\%$

22.(4) **(1)** $[Ni(CN)_4]^{2-} : Ni^{2+} \rightarrow 3d^8$

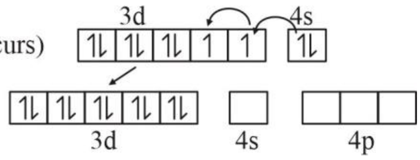
CN^-(SFL→ pairing occurs)



No unpaired e^- : Diamagnetic

(2) $[Ni(CO)_4] : Ni \rightarrow 3d^8 4s^2$

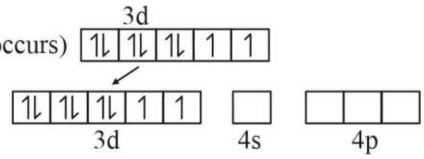
CO (SFL→ pairing occurs)



No unpaired e^- : Diamagnetic

(3) $[Ni(Cl)_4]^{2-} : Ni^{2+} \rightarrow 3d^8$

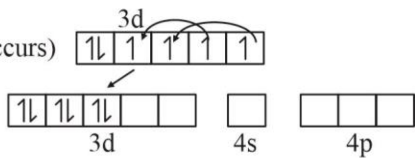
Cl^-(WFL→ No pairing occurs)



Two unpaired e^- : Paramagnetic

(4) $[Fe(CN)_6]^{4-} : Fe^{2+} \rightarrow 3d^6$

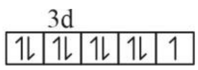
CN^- (SFL→ pairing occurs)



No unpaired e^- : Diamagnetic

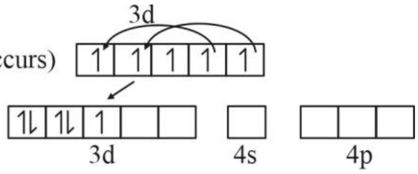
(5) $[Cu(NH_3)_4]^{2+} : Cu^{2+} \rightarrow 3d^9$

NH_3 (WFL→ No pairing occurs)



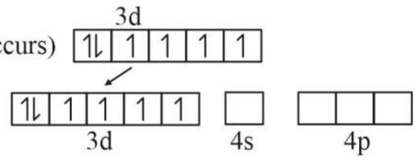
One unpaired e^- : Paramagnetic

(6) $[Fe(CN)_6]^{3-} : Fe^{3+} \rightarrow 3d^5$

CN^- (SFL→ pairing occurs) 

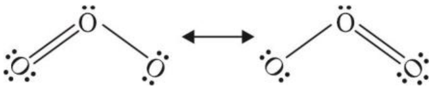
One unpaired e^- : Paramagnetic

(7) $[Fe(H_2O)_6]^{2+}$: $Fe^{2+} \rightarrow 3d^6$

H_2O (WFL→ No pairing occurs) 

Four unpaired e^- : Paramagnetic

23.(6) Structure of ozone (O_3)



Total number of lone pairs = 6

24.(2) $V^{3+}: 3d^2$ [1 | 1 | | |] → No. of unpaired $e^- = 2$

$Cr^{3+}: 3d^3$ [1 | 1 | 1 | |] → No. of unpaired $e^- = 3$

$Fe^{2+}: 3d^6$ [1l | 1 | 1 | 1 | 1] → No. of unpaired $e^- = 4$

$Ni^{3+}: 3d^7$ [1l | 1l | 1 | 1 | 1] → No. of unpaired $e^- = 3$

Spin-only magnetic moment, $\mu = \sqrt{n(n+2)}$ BM

Where n is no. of unpaired e^-

So, Cr^{3+} & Ni^{3+} have same no. of unpaired e^- & hence, same value of μ.

25.(9079) $NH_4OH + H^+ \rightleftharpoons NH_4^+ + H_2O\ ;\ K_{eq} = \frac{K_b}{K_w}$

$t = 0$	0.1	0.02	0.1	-
$t = t'$	0.08	-	0.12	-
$t = t_{eq}$	$0.08 + x$	x	$0.12 - x$	
	$\simeq 0.08$		$\simeq 0.12$	

So, $\frac{0.12}{0.08 \times x} = \frac{K_b}{K_w}$

$x = [H^+] = \frac{3}{2} \times \frac{K_w}{K_b}$

$pH = (-\log 3) - (-\log 2) - \log K_w - (-\log K_b)$

$pH = \log 2 - \log 3 + pK_w - pK_b$

$pH = 0.301 - 0.477 + 14 - 4.745$

$pH = 9.079 = 9079 \times 10^{-3} = 9079$

26.(41500)

$\pi = CRT$

$\frac{\pi}{C} = RT = \text{constant}$

27.(10) $E_{cell} = E^0_{cell} - \frac{0.059}{n}\log Q$

$E^0_{cell} = E^0_{cathode} - E^0_{anode} = 0.771V - 0.000V = 0.771V$

RHR : $Fe^{3+} + e^- \longrightarrow Fe^{2+}$

OHR : $\frac{1}{2}H_2 \longrightarrow H^+ + e^-$

$Fe^{3+} + \frac{1}{2}H_2 \longrightarrow Fe^{2+} + H^+$

$Q = \frac{[Fe^{2+}][H^+]}{[Fe^{3+}]\,P_{H_2}^{1/2}}$

$[H^+] = 1M,\ P_{H_2} = 1atm,\ n = 1$

$Q = \frac{[Fe^{2+}]}{[Fe^{3+}]}$

$E_{cell} = 0.712V$ [given] $\quad\therefore\ 0.712 = 0.771 - \frac{0.059}{1}\log\frac{[Fe^{2+}]}{[Fe^{3+}]}$

$\frac{[Fe^{2+}]}{[Fe^{3+}]} = 10$

28.(60) For first order

$t_{1/2} \propto [A_0]^0$

For 75% completion

$t = 2t_{1/2} = 2(30) = 60\,min$

29.(360) Energy released = 1800 KJ

Energy utilized in sports = 900 KJ

Excess energy $= 1800 - 900 = 900\,KJ$

Moles of water vaporized using excess energy $= \frac{900}{45} = 20\,moles$

Mass of water $= 18 \times 20 = 360\,g$

30.(12) Molarity of acid $= \frac{1.21\times10^3}{24.2}\,mol/L$

$M_{acid} = 50\,mol/L$

$(n_f)_{acid} = 1$ (Monobasic acid)

$(meq)_{NaOH} = (n_f \times M \times V)_{NaOH}$

$(n_f)_{NaOH} = 1$ (Monoacidic base)

$M_{NaOH} = 0.24\,mol/L,\ V_{NaOH} = 25\,mL$

So, $(meq)_{NaOH} = 1 \times 0.24 \times 25 = 6$

For complete neutralization

$(meq)_{acid} = (meq)_{NaOH}$

$1 \times 50 \times V_{acid} = 6$

$V_{acid} = \frac{6}{50}\,mL$ = 0.12 mL $= 12\times10^{-2}\,mL = 12$

MATHEMATICS

SECTION - 1

1.(2) $\Delta=\begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}=a(15a^2+31a+37)\neq 0\ \forall a\in R-\{0\}$

$\Rightarrow$ Unique solution for all $x\in R-\{0\}$

2.(3) $\lim_{n\to\infty}\dfrac{\sum_{r=1}^{n}((3r-2)+(3r-1-3r)}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$

$\lim_{n\to\infty}\dfrac{\sum_{r=1}^{n}3(r-1)}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$

$=\lim_{n\to\infty}\dfrac{3\frac{n(n-1)}{2}\left(\sqrt{2n^4+4n+3}+\sqrt{n^4+5n+4}\right)}{(2n^4+4n+3)-(n^4+5n+4)}=\frac{3}{2}\left(\sqrt{2}+1\right)$

3.(2) Let number of observations is n

$(10.2)n=10n-8+12$

$\Rightarrow (10.2)n=10n+4 \Rightarrow n=20$

For earlier observation set

$\frac{\Sigma x_i^2}{20}-(10)^2=4$

$\Sigma x_i^2=(104)(20)=2080$

After change

$(\Sigma x_i^2)_{new}=2080-8^2+12^2=2160$

New variance $=\frac{2160}{20}-(10.2)^2=108-(10.2)^2=3.96$

4.(4) Put $z=x+iy$ we get $(x-2)^2+(y-3)^2-(x-3)^2-(y-4)^2=2$

$\Rightarrow 2x+2y=14 \Rightarrow \frac{x}{7}+\frac{y}{7}=1$

5.(1) $\frac{dy}{dx}-\frac{y}{x}=y^3(1+\ln x)$

$\frac{1}{y^3}\frac{dy}{dx}-\frac{1}{x}\frac{1}{y^2}=(1+\ln x)$

$\frac{1}{y^2}=t \Rightarrow \frac{-2}{y^3}\frac{dy}{dx}=\frac{dt}{dx}$ $\quad\therefore \frac{-1}{2}\frac{dt}{dx}-\frac{t}{x}=(1+\ln x)$

$\frac{dt}{dx}+\frac{2t}{x}=-2(1+\ln x)$

$tx^2=\int -2(1+\ln x)x^2dx$

$$\frac{x^2}{y^2} = -2\left[\frac{x^3}{3}(1+\ln x) - \frac{x^3}{9}\right] + c \qquad \ldots\text{(i)}$$

$$y(1) = 3 \Rightarrow \frac{1}{9} = -2\left(\frac{1}{3} - \frac{1}{9}\right) + c \qquad \therefore\ c = \frac{5}{9}$$

$$\Rightarrow \frac{x^2}{y^2} = \frac{5}{9} - \frac{2}{3}x^3(1+\log_e x) + \frac{2x^3}{9} \quad \Rightarrow \frac{9x^2}{y^2} = 5 - 6x^3(1+\log_e x) + 2x^3$$

$$\Rightarrow \frac{y^2}{9} = \frac{x^2}{5 - 4x^3 - 6x^3\log_e x} = \frac{x^2}{5 - 2x^3(2+\log_e x^3)}$$

6.(3) $y = (x+1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)$

Multiplying and dividing by $(x-1)$ we get $y = \dfrac{x^{32}-1}{x-1}$

At $x = -1,\ y = 0$

$y(x-1) = x^{32} - 1$

Differentiate on both side

$y'(x-1) + y = 32x^{31} \quad \ldots\text{(i)}$

At $x = -1$

$y'(-1) = 16$

Diff. (i) on both side

$y''(x-1) + y' + y' = 32 \times 31x^{30}$

Substitute $x = -1$

$y''(-1) = -480 \qquad \Rightarrow\ y' - y''$ at $x = -1$ is 496

7.(1) $\vec{b} = \lambda\vec{a} + \mu\vec{j} = (\lambda(-\hat{i} + 2\hat{j} + \hat{k}) + \mu\hat{j})$

$\vec{b}\cdot\vec{a} = 0$

$(\lambda\vec{a} + \mu\hat{j})\cdot\vec{a} = 0$

$6\lambda + 2\mu = 0$

$\mu = -3\lambda$

$\vec{b} = \lambda(\vec{a} - 3\hat{j}) = \lambda(-\hat{i} - \hat{j} + \hat{k})$

$\lambda = \pm\sqrt{2} \qquad \because\ |\vec{b}| = |\vec{a}|$

$\vec{a}\cdot\vec{c} = 6,\ \vec{b}\cdot\vec{c} = -6\lambda$

Now $(3\vec{a} + \vec{b}\sqrt{2})\cdot\dfrac{\vec{c}}{|\vec{c}|} = \dfrac{3\vec{a}\cdot\vec{c} + \sqrt{2}\vec{b}\cdot\vec{c}}{|\vec{c}|} = \dfrac{3(6) - 6\sqrt{2}\lambda}{\sqrt{50}} = \dfrac{18 \pm 12}{5\sqrt{2}} = 3\sqrt{2}$ or $\dfrac{3\sqrt{2}}{5}$

8.(4) Circle $x^2 + y^2 - 2x = 0$ passing $A(\alpha, 0)$ & $B(1, \beta)$

$\Rightarrow \alpha^2 - 2\alpha = 0$ & $\beta^2 - 1 = 0 \ \because\ m_{AB} \neq -1$

$ax + by = 0$ pass $A(0,0)$ & $B(1,1)$

$\Rightarrow AB$ is $x - y = 0$

Centre of circle whose diameter is AB is $C\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and radius $= \dfrac{1}{\sqrt{2}}$

Image of C w.r.t. line $x+y+2=0$ is $C'\left(\frac{-5}{2},\frac{-5}{2}\right)$

$\Rightarrow$ Required circle is $\left(x+\frac{5}{2}\right)^2+\left(y+\frac{5}{2}\right)^2=\left(\frac{1}{\sqrt{2}}\right)^2$

$\Rightarrow x^2+y^2+5x+5y+12=0$

9.(4) $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\frac{\vec{b}-\vec{c}}{2}$ $\Rightarrow \vec{a}\cdot\vec{c}=\frac{1}{2}$ & $\vec{a}\cdot\vec{b}=\frac{1}{2}$

Given $\vec{b}\cdot\vec{d}=\vec{a}\cdot\vec{b}$ & $\vec{b}\cdot\vec{c}=0$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{b})=(\vec{b}\cdot\vec{d})(\vec{a}\cdot\vec{c})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)-(\vec{a}\cdot\vec{d})(0)=\frac{1}{4}$

10.(4) $g(x)=f(-x)-f(x)=\frac{1}{1-e^x}-\frac{e^x}{e^x-1}=\frac{1+e^x}{1-e^x}=\frac{e^x-1+2}{1-e^x}=-1+\frac{2}{1-e^x}$

$g'(x)=\frac{2e^x}{(1-e^x)^2}>0\ \forall x\in(0,1)$

Clearly $g(x)$ is continuous and strictly increasing in (0, 1) hence $g(x)$ is one-one

11.(3) $(p\wedge\sim q)\to(p\to\sim q)=(p\wedge\sim q)\to(\sim p\vee\sim q)=\sim(p\wedge\sim q)\vee(\sim p\vee\sim q)$

$=\sim p\vee q\vee(\sim p\vee\sim q)=\sim p\vee T=T$ (Tautology)

12.(2) For $x>2$

$f(x)=\int_0^2 e^{x-t}dt=e^x\left(-e^{-t}\right)\Big|_0^2=e^x(1-e^{-2})$

For $x<0$

$f(x)=\int_0^2 e^{t-x}dt=e^{-x}e^t\,\Big|_0^2=e^{-x}(e^2-1)$

For $0\le x\le 2$

$f(x)=\int_0^x e^{x-t}dt+\int_x^2 e^{t-x}dt=-e^xe^{-t}\,\Big|_0^x+e^{-x}e^t\,\Big|_x^2$

$\Rightarrow -e^x(e^{-x}-1)+e^{-x}(e^2-e^x)$ $\Rightarrow -1+e^x+e^{2-x}-1=e^{2-x}+e^x-2$

$$f(x)\begin{cases}e^x(1-e^{-2})\ ;\ x>2\\ e^{2-x}+e^x-2\ ;\ 0\le x\le 2\\ e^{-x}(e^2-1)\ ;\ x<0\end{cases}$$

$f(x)$ is decreasing function for $x<0$ increasing function for $x\ge 2$

Now for $0\le x\le 2,\ f(x)$ is minimum at $x=1$, by using A.M. $\ge$ G.M.

$\therefore$ Minimum value of $f(x)$ is $f(1)=2(e-1)$

13.(4) Let P is $(2\lambda+1,\ 3+\lambda,\ 2+2\lambda)$ and Q is $(2+\mu,\ 2+2\mu,\ 3+3\mu)$

$\overline{PQ}$ and $\hat{i}-\hat{j}-2\hat{k}$ are collinear vectors $\Rightarrow \frac{2\lambda-\mu-1}{1}=\frac{\lambda-2\mu+1}{-1}=\frac{2\lambda-3\mu-1}{-2}$

$\Rightarrow \lambda=\mu=3 \Rightarrow P(7,6,8)$ & $Q(5,8,12)$ $\Rightarrow PQ=\sqrt{4+4+16}=2\sqrt{6}$

14.(3) $|A| = \frac{1}{\log x \log y \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & 2\log y & \log z \\ \log x & \log y & 3\log z \end{vmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$|A| = 2$$

$$|adj(adjA^2)| = |A|^8 = 2^8$$

15.(3) $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)}dx$

$$x^2 = t$$

$$2xdx = dt$$

$$\int \frac{dt}{(t+1)(t+3)}$$

$$= \frac{1}{2}\int \frac{(t+3)-(t+1)}{(t+1)(t+3)}dt = \frac{1}{2}[\ln|t+1| - \ln|t+3|] + \frac{C}{2} = \frac{1}{2}[\ln|x^2+1| - \ln|x^2+3|] + \frac{C}{2}$$

$$\because\ f(3) = \frac{1}{2}[\ln 5 - \ln 6] \qquad \therefore\ \frac{1}{2}[\ln 5 - \ln 6] = \frac{1}{2}[\ln 10 - \ln 12] + \frac{C}{2} \Rightarrow C = 0$$

$$\therefore\ f(x) = \frac{1}{2}[\ln|x^2+1| - \ln|x^2+3|]$$

$$f(4) = \frac{1}{2}[\ln 17 - \ln 19]$$

16.(1) Tangent of slope m of $y^2 = \frac{x}{2}$ is $y = mx + \frac{1}{8m}$

Tangent of slope m of $y^2 = x - 1$ is $y = m(x-1) + \frac{1}{4m}$

For common tangent $C_1 = C_2 \Rightarrow \frac{1}{8m} = -m + \frac{1}{4m}$

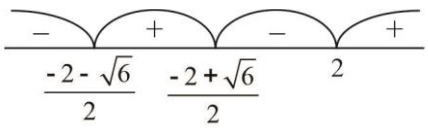
$\Rightarrow\ m = \frac{1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}$ Given $m > 0$ $\quad \Rightarrow\ y = \frac{x}{2\sqrt{2}} + \frac{2\sqrt{2}}{8}$ is common tangent

$\Rightarrow\ x - 2\sqrt{2}y + 1 = 0$

Now perpendicular distance from $(6, -2\sqrt{2})$ is 5

17.(4) $f'(x) = 8x^3 - 36x + 8 = 4(x-2)(2x^2+4x-1)$

Local max exist at $x = \frac{-2+\sqrt{6}}{2} \Rightarrow 2x^2 = 1 - 4x$

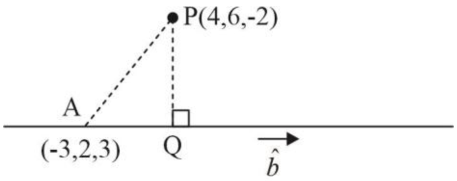


$$2x^4 - 18x^2 + 8x + 12 = \frac{1}{2}(1-4x)^2 - 9(1-4x) + 8x + 12$$

$$= \frac{7}{2} + 8x^2 + 40x = \frac{7}{2} + 4(1-4x) + 40x = \frac{15}{2} + 24x = \frac{15}{2} + 12\left(\sqrt{6}-2\right) = 12\sqrt{6} - \frac{33}{2}$$

18.(1) $PQ = \left|\overline{AP} \times \hat{b}\right|$

$$= \frac{\left|(7\hat{i} + 4\hat{j} - 5\hat{k}) \times (3\hat{i} + 3\hat{j} - \hat{k})\right|}{\sqrt{19}} = \frac{\sqrt{266}}{\sqrt{19}} = \sqrt{14}$$



19.(4) $a_r = {}^{10}C_{10-r}$

$$\sum_{r=1}^{10} r^3 \left(\frac{{}^{10}C_{10-r}}{{}^{10}C_{11-r}}\right)^2 = \sum_{r=1}^{10} r^3 \left(\frac{10!}{r!(10-r)!} \cdot \frac{(11-r)!(r-1)!}{10!}\right)^2$$

$$= \sum_{r=1}^{10} r^3 \left(\frac{11-r}{r}\right)^2 = \sum_{r=1}^{10} r(11-r)^2 = \sum_{r=1}^{10} r^2(11-r) = 11\sum_{r=1}^{10} r^2 - \sum_{r=1}^{10} r^3$$

$$= 11\left(\frac{10 \cdot 11 \cdot 21}{6}\right) - \left(\frac{10 \cdot 11}{2}\right)^2 = (11)^2 35 - (11)^2 \cdot 25 = (11)^2 \times 10 = 1210$$

20.(1) $M = 33 \times 33 = 1089$

$x(66 - x) \geq 605$

$x^2 - 66x + 605 \leq 0$

$x \in [11, 55]$

Favourable set of values of x for event $A = \{12, 15, 18, \ldots.., 54\}$

$$P(A) = \frac{15}{45} = \frac{1}{3}$$

SECTION – 2

21.(495) $a = A + 6d$

$b = A + 1 + 8d$

$c = A + 2 + 16d$

$$\begin{vmatrix} A+6d & 7 & 1 \\ 2A+2+16d & 17 & 1 \\ A+2+16d & 17 & 1 \end{vmatrix} = -70 \text{ Apply } R_2 - R_1,\ R_3 - R_1 \Rightarrow \begin{vmatrix} A+6d & 7 & 1 \\ A+10d+2 & 10 & 0 \\ 2+10d & 10 & 0 \end{vmatrix} = -70 \Rightarrow A = -7$$

Given $a = 29 \Rightarrow 29 = -7 + 6d \Rightarrow d = 6$

$\Rightarrow c - a - 6 = -A + 1 + 2d = 20$

$$S_{20} = \frac{20}{2}\left[2(20) + (20-1)\frac{1}{2}\right] = 495$$

22.(9) Equation of plane is $(x - 2y - z - 5) + \lambda(x + y + 3z - 5) = 0$

Normal vector $\vec{n} = (\lambda + 1)\hat{i} + (\lambda - 2)\hat{j} + (3\lambda - 1)\hat{k}$

This plane is parallel to $\vec{n}_3 \times \vec{n}_4 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = -5\hat{i} + 3\hat{j} + \hat{k} \quad \Rightarrow \vec{n} \cdot (\vec{n}_3 \times \vec{n}_4) = 0 \Rightarrow \lambda = 12$

Hence required plane is $13x + 10y + 35z = 65 \Rightarrow (a, b, c)$ is $(13, 10, 35)$

Distance from (13, 10, 35) to the plane $2x + 2y - z + 16 = 0$

$$\frac{|26 + 20 - 35 + 16|}{\sqrt{4 + 4 + 1}} = 9$$

23.(120) Type — Numbers

Type	Numbers
$5k$	5, 10, 15, 20, 25
$5k+1$	1, 6, 11, 16, 21
$5k+2$	2, 7, 12, 17, 22
$5k+3$	3, 8, 13, 18, 23
$5k+4$	4, 9, 14, 19, 24

(x, y) can be selected as

1 of $(5k+1)$ and 1 of $(5k+4) = 2\times 25 = 50$

1 of $(5k+2)$ and 1 of $(5k+3) = 2\times 25 = 50$

Both of the type $5k = 20$

Total = 120

24.(1080) $\left(2x+\frac{1}{x^7}+3x^2\right)^5 = \frac{(1+2x^8+3x^9)^5}{x^{35}}$

Coefficient of x^{35} in $(1+x^8(2+3x))^5$

Coefficient of x^{35} in ${}^5C_4(x^8(2+3x))^4$

$\Rightarrow\ {}^5C_4 \times$ coefficient of x^3 in $(2+3x)^4$ $\qquad \Rightarrow\ {}^5C_4 \times {}^4C_3(2)^1(3)^3 = 1080$

25.(2) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$

For $-1<x<0$

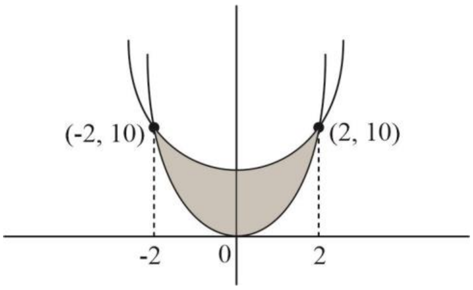
$2\tan^{-1}x+2\tan^{-1}x+\pi=\frac{\pi}{3} \quad \Rightarrow\ 4\tan^{-1}x=-\frac{2\pi}{3} \quad \Rightarrow\ x=-\frac{1}{\sqrt{3}}$

For $0<x<1$

$4\tan^{-1}x=\frac{\pi}{3} \quad \Rightarrow\ x=\tan\frac{\pi}{12}=2-\sqrt{3}$

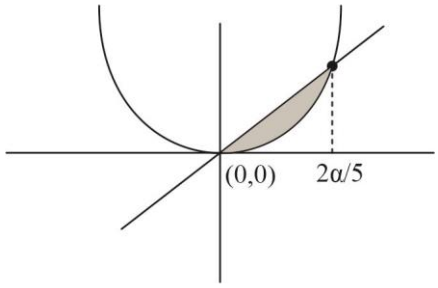
Sum $=2-\sqrt{3}-\frac{1}{\sqrt{3}}=2-\frac{4}{\sqrt{3}}$

26.(600) A_1 is area enclosed by $y=\frac{5x^2}{2}$ & $y=6+x^2$



$A_1=\int_{-2}^{2}(6+x^2)-\frac{5x^2}{2}dx=16$

$A_2=\int_{0}^{\frac{2\alpha}{5}}\left(\alpha x-\frac{5x^2}{2}\right)dx=\frac{2\alpha^3}{75}$



$A_2=A_1 \Rightarrow \frac{2\alpha^3}{75}=16 \Rightarrow \alpha^3=600$

27.(25) $\log_2(9^{2\alpha-4}+13)-\log_2\left(3^{2\alpha-4}\cdot\frac{5}{2}+1\right)=2$

$\frac{9^{2\alpha-4}+13}{3^{2\alpha-4}\cdot\frac{5}{2}+1}=4$

Let $3^{2\alpha-4}=t$

$t^2+13=10t+4$

$t^2-10t+9=0$

$\therefore\ t=9,1$

$\Rightarrow\ \alpha=3,2$

$\Rightarrow\ \Sigma\alpha=3+2=5$ and $\Sigma(\alpha+1)^2=(3+1)^2+(2+1)^2=25$

Roots of $x^2-50x+25\beta=0$ are real

$\Rightarrow D\le 0 \Rightarrow 2500-100\beta\ge 0 \Rightarrow \beta\le 25$

$\Rightarrow\ \beta_{max}=25$

28.(43) Out of the given numbers one is 3k type and 3 of $3k+1$ type and remaining three are $3k+2$ type

Number of subsets with 1 element = 1

1 of 3k type

Number of subsets with 2 elements

1 of $(3k+1)$ type + 1 of $(3k+2)$ type = 9

Number of subsets with 3 elements

1 of $3k$ type + 1 of $(3k+1)$ type +1 of $(3k+2)$ type = 9

3 of $(3k+1)$ type = 1

3 of $(3k+2)$ type = 1

Number of subsets with 4 elements

1 of $3k$ type + 3 of $(3k+1)$ type = 1

1 of $3k$ type + 3 of $(3k+2)$ type = 1

2 of $(3k+1)$ type + 2 of $(3k+2)$ = 9

Number of subsets with 5 elements

1 of $3k$ type + 2 of $(3k+1)$ type + 2 of $(3k+2)$ type = 9

Number of subsets with 6 elements

3 of $3k+1$ type + 3 of $3k+2$ type = 1

The set itself = 1

Total = 43

29.(2039) $fog(x)=a\,g(x)-3=a(x^b+c)-3$

$$\Rightarrow\ x^b+c=\frac{y+3}{a}\ \Rightarrow\ x=\left(\frac{y+3-ac}{a}\right)^{\frac{1}{b}}$$

$$\Rightarrow\ (fog)^{-1}(x)=\left(\frac{x+3-ac}{a}\right)^{\frac{1}{b}}=\left(\frac{x-7}{2}\right)^{\frac{1}{3}}$$

$\Rightarrow a=2, b=3, c=5$

$\Rightarrow\ f(x)=2x-3\ \&\ g(x)=x^3+5$

$fog(ac)=fog(10)=f(1005)=2007$

$gof(b)=gof(3)=g(3)=32$

$\Rightarrow\ fog(ac)+gof(b)=2039$

30.(216) $a=6,\ e=\frac{\sqrt{5}}{2}$

$$b^2=a^2e^2-a^2=\left(3\sqrt{5}\right)^2-(6)^2=9$$

Normal at $P(6\sec\theta,\ 3\tan\theta)$

$$\frac{6x}{\sec\theta}+\frac{3y}{\tan\theta}=36+9$$

Slope of normal $=-2\sin\theta=-\sqrt{2}$

$$\Rightarrow\ \sin\theta=\frac{1}{\sqrt{2}}\ \Rightarrow\ \theta=\frac{\pi}{4}$$

Normal cut the y-axis at $Q\left(0,15\tan\frac{\pi}{4}\right)$

$$d^2=(PQ)^2=\left(6\sec\frac{\pi}{4}-0\right)^2+\left(3\tan\frac{\pi}{4}-15\tan\frac{\pi}{4}\right)^2=72+144=216$$