Indian Olympiad Qualifier in Mathematics (IOQM - 2023) Solutions

1.(22) $n \in [1,1000], n \in N$

$x_n \in \{\sqrt{4x+1}, \sqrt{4x+2}, ... \sqrt{4x+1000}\}$

We can observe that the any element of x_n will be integer if $4n+k$ is a perfect square ($k \in [1,1000], k \in N$). We also can observe that difference of consecutive perfect squares increases.

So for $a = \max\{M_n : 1 \le n \le 1000\}$ n should be least (n = 1) & for

$b = \min\{M_n : 1 \le n \le 1000\}^n$ should be max $(n = 1000)$

$\therefore \quad a = 29 \rightarrow \{x_n \in \{\underset{2.2}{\underset{\downarrow}{\sqrt{5}}}, ... \underset{31.6}{\underset{\downarrow}{\sqrt{1004}}},\}$

$b = 7 \rightarrow \{\underset{63.2}{\underset{\downarrow}{\sqrt{4001}}}, ... \underset{70.7}{\underset{\downarrow}{\sqrt{5000}}},\}$

$\therefore \quad a - b = 22$

2.(54) $\log_a^b + 6\log_b^a = 5, \qquad (a,b) \in N \qquad 2 \le a, b \le 2023$

Let $\quad \log_a^b = t, \qquad a \,\&\, b \ne 1$

$t + \dfrac{6}{t} = S$

$\Rightarrow \; t^2 - 5t + 6 = 0$

$\Rightarrow \; t = 2, 3$

$\Rightarrow \; \log_a^b = 2, 3$

$\Rightarrow \; b = a^2, a^3$

Common solution $\rightarrow$ (1, 1)

Case I

$b = a^2$

Possibilities for a = 1, 2, 3 ..... 44 $\{\sqrt{2023} = 44.97\}$

$\therefore \quad (a,b) \rightarrow 44$ solution

Case II

$b = a^3$

Possibilities for a = 1, 2, 3 12

$\therefore$ (a, b) = 12 Solution

$\therefore$ Total solution = 44 + 12 – 2 = 54

3.(296) $\dfrac{16}{37} < \dfrac{\alpha}{\beta} < \dfrac{7}{16}$

$\dfrac{256}{592} < \dfrac{\alpha}{\beta} < \dfrac{259}{592}$

$\therefore \quad \dfrac{\alpha}{\beta}$ can be $\rightarrow \; \dfrac{257}{592}$ or $\dfrac{258}{592}$

$\dfrac{258}{592} = \dfrac{129}{296} \qquad 258 = 2 \times 3 \times 43$

$592 = 2^4 \times 37$

$\therefore \quad$ Smallest $\beta = 296$

4.(7) $x^4 = (x-1)(y^3-23)-1$

$x^4+1 = (x-1)(y^3-23)$

$y^3 - 23 = \dfrac{x^4+1}{x-1}$ ...(i)

$y^3 - 23 = \dfrac{x^4-1}{x-1} + \dfrac{2}{x-1}$

$y^3 - 23 = (x^2+1)(x+1) + \dfrac{2}{x-1}$

$\therefore$ $x-1 = 1$ or 2

$x = 2$ or 3

If $x = 2$,

than $y^3 = 23+17 = 40$, no solution

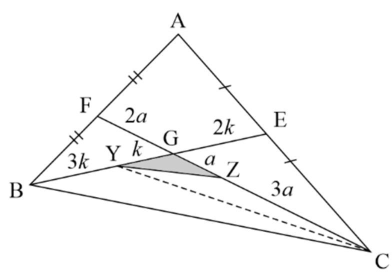
If $x = 3$, than $y^3 = 23+41 = 64$

$\therefore$ $y = 4$

$\therefore$ $(x, y) = (3, 4) \rightarrow$ only solution

$x + y = 7$

5.(10)



Let- $BE = 6k$ Join YC

$FC = 6a$

$3Y = 3k$ $FG = 2a$

$YG = k$ $GZ = a$

$GE = 2k$ $ZC = 3a$

$ar(\Delta ABC) = 240$

$ar(\Delta BEC) = \dfrac{240}{2} = 120$

$ar(\Delta BGC) = \dfrac{1}{3} \times 120 = 40$

$ar(\Delta YGC) = \dfrac{1}{4} \times 40 = 10$

6.(16) Let number of sides of polygon = a, $a \neq 1, 2$, $a \in N$

Interior angle $\Rightarrow \dfrac{(a-2).180}{a} = n$

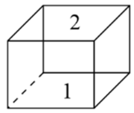
$n = 2 \times \left(\dfrac{90a-180}{a}\right) = 2 \times \left(90 - \dfrac{180}{a}\right)$

$\therefore$ a is a factor of 180

$180 = 2^2 \times 3^2 \times 5^1 \rightarrow 3 \times 3 \times 2 = 18$

But $a \neq 1, 2$ $\therefore$ Number of values of $n = 18 - 2 = 16$

7. (24)



Pairing of other number's (3, 4, 5, 6)

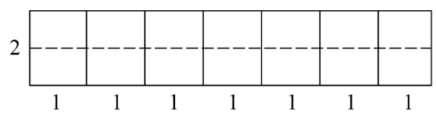
Possible by $\frac{4!}{2!2!2!}$ Ways $=\frac{24}{8}=3$

$\left(\begin{smallmatrix}34\\56\end{smallmatrix}\ or\ \begin{smallmatrix}35\\46\end{smallmatrix}\ or\ \begin{smallmatrix}36\\45\end{smallmatrix}\right)$

$\therefore\ 3\times 2\times 2\times 2=24$
coloring for each pair

Paring and arrangement of numbers.

8.(31) So sum of square of digits = $3^2+9^2+2^2=94$



Without a 2 × 2 tile 1 way

With a 2 × 1 tile 30 ways

Total number of ways = 31.

9.(17) If $a=1$, $b=2$, $c=3,5,7,13$

$b=3$, $c=2,5,7$

$b=5$, $c=2,3$

$b=7$, $c=2,3$

$b=11$, $c=2$

$b=13$, $c=2$

Here, of 14 triples.

If $a=2$, $b=1$, $c=15$

$a=3$, $b=1$, $c=10$

$a=5$, $b=1$, $c=6$

$\Rightarrow$ Total number of triples (a, b, c) = 14 + 3 =17

10.(51) $a_0 = 1, a_1 = -4$

$a_{n+2} = -4 - a_{n+1} - 7a_n$

Claim: $a_n^2 - a_{n+1}a_{n-1} = 7^n$

For all $n \geq 1$

Obviously thus for n = 1,

Say it is true for $n \leq k$

$ak^2 - a_{k+1}9_{k-1} = 7^k$

To prove: $a_{k+1}{}^2 - a_{k+2}a_k = 7^{k+1}$

Now $a_{k+2} = -4a_{k+1} - 79_k$

LHS $= 9k+1^2 + (49_{k+1} + 79_k)\ a_k$

$= a_{k+1}^2 + (49_{k+1} + 79_k)a_k$

$= a_{k+1}^2 + 49_{k+1}9k + 7(a_k^2 - 9_{k+1}9_{k+1}) + 79_{k+1}9_{k-1}$

$= 9_{k+1}\left(a_{k+1} + 49_k + 79_{k-1}\right) + 7\left(a_k^2 - a_{k+1}a_{k-1}\right)$

$= 9_{k+1} \cdot 0 + 7 \cdot 7^k = 7^{k+1}$ Proved

Therefore $a_{50}^2 - a_{49}a_{51} = (7)^{50}$

Number of divisors = 51

11.(44) $m^2 = 4n^2 - 5n + 16$

$$m^2 = 4\left(n^2 - \frac{5}{4}n\right) + 16$$

$$m^2 = 4\left(\left(n - \frac{5}{8}\right)^2 - \frac{25}{64}\right) + 16$$

$$m^2 = 4\left(n - \frac{5}{8}\right)^2 - \frac{25}{16} + 16$$

$16m^2 = (8n-5)^2 + 231$

$(4m)^2 + (8m-5)^2 = 231$

Plug in $4m - 8n + 5 = -1$ and $4m + 8n - 5 = -21$

$m = -29, n = 15$; $\quad |m-n|_{\max} = 44.$

12.(18) $p(x)=x^3+ax^2+bx+c$

$p_1^3+p_2^3+p_3^3=3p_1p_2p_3$.

$\Rightarrow$ Either $p_1+p_2+p_3=0$ *or* $p_1=p_2=p_3$

Now $p_1+p_2+p_3=36+14a+6b+3c$ which is odd and therefore cannot be O.

Now

$$\Rightarrow \quad p_1=p_2=p_3$$

$$1+a+b+c=8+4a+2b+c=27+9a+3b+c$$

$$\Rightarrow \quad 3a+b=-7$$

$$5a+b=-19$$

$$2a=-12 \Rightarrow a=-6;\ b=11$$

$$p_2+2p_1-3p_0$$

$$(8+4a+2b+c)+2(1+a+b+c)-3c$$

$$10+6a+4b=18$$

13. $r_1=\dfrac{\Delta}{s-a}=\dfrac{21}{2}$

$r_2=\dfrac{\Delta}{s-b}=12$

$r_3=\dfrac{\Delta}{s-c}=14$

$\dfrac{s-b}{s-a}$ $\quad s-a=8k \quad a+b+c=42k$

$\quad s-b=7k \quad 42k=21$

$\dfrac{s-c}{s-b}$ $\quad s-c=6k \quad k=\dfrac{1}{2}$

Also $\quad r_1\ r_1r_2+r_2r_3+r_3r_1=s^2$

$$s=\sqrt{\frac{21}{2}\times12+12\times14+14\times\frac{21}{2}}=\sqrt{126+168+147}=\sqrt{441}=21$$

$$s-a=4\rightarrow a=21-4=17$$

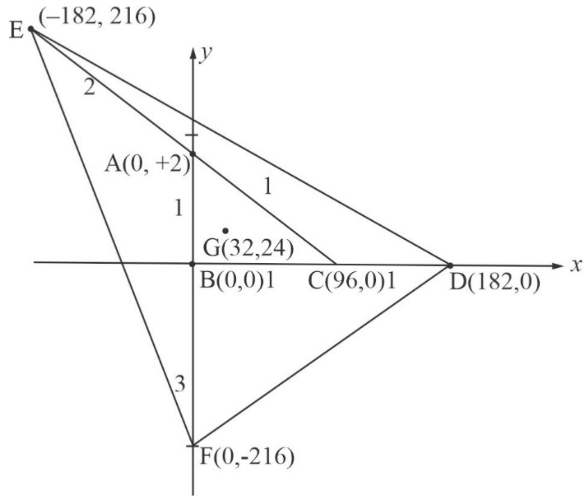
$$s-b=\frac{7}{2}\rightarrow b=21-\frac{7}{2}=\frac{35}{2}=17.5$$

$$s-c=3\rightarrow c=21-3=18$$

$$\sqrt{p+q+r}=\sqrt{21+17\times\frac{35}{2}\times18+18\times17+17\times\frac{35}{2}\times18}$$

$$=\sqrt{21+297.5+315+306+5355}=\sqrt{6294.5}\approx 79.33$$

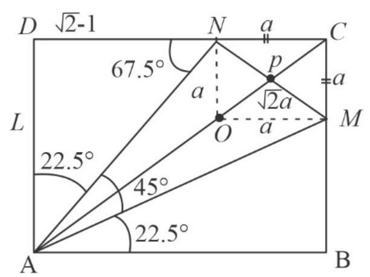
14.(40) 'B' is centroid of ΔDEF .



$\therefore \quad k(0,0)\,\&\,G(32,34)$

$\therefore \quad GK=\sqrt{32^2+34^2}=40$.

15.(03) WHOG, Let CM = CN



$a(2+\sqrt{2})=2$

$OP=\dfrac{a}{\sqrt{2}}$

$OA=AC-OC$

$=\sqrt{2}-a\sqrt{2}$

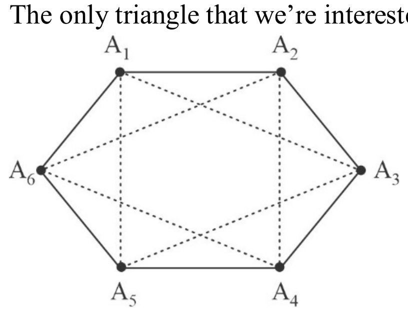
$=\sqrt{2}-2\sqrt{2}+2$

$AO=2-\sqrt{2}=a$

$$\therefore \quad \left(\frac{OP}{OA}\right)^2=\left(\frac{\frac{a}{\sqrt{2}}}{a}\right)^2=\frac{1}{2}=\frac{m}{n}$$

$\Rightarrow \quad m+n=3$

16.(94) Energy triangle having a side common with the hexagon will have at least one of its sides rod. The only triangle that we're interested in is these which do not shown a side with the hexagon.



$8.(2^3-1)^2=49.8=392=N$

So sum of square of digits $=3^2+9^2+2^2=94$

17.(66) Say $d = r$

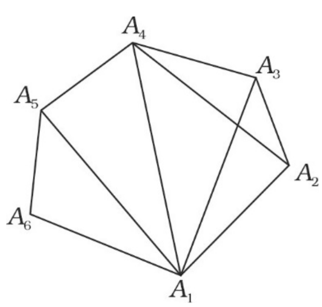
So, Average $\dfrac{= \displaystyle\sum_{r=7}^{98} {}^{r-1}C_3(99-r).\, r}{= \displaystyle\sum_{r=4}^{98} {}^{r-1}C_3(99-x)}$

$$\frac{\displaystyle\sum_{r=4}^{98} \frac{1}{6}\ r(r-1)\,(r-2)\ r(r-3)\ \ (99-r)}{\displaystyle\sum_{r=4}^{98} \frac{1}{6}\ (r-1)\,(r-2)\,r(r-3)\ \ (99-r)}$$

$$\frac{\frac{1}{6}\cdot 99\displaystyle\sum_{r=4}^{98} r(r-1)(r-2)(r-3) - \frac{1}{6}\sum_{r=4}^{98} r.r(r-1)(r-2)(r-3)}{\frac{99}{6}\displaystyle\sum_{r=4}^{98}(r-1)\ (r-2)\,(r-3) - \frac{1}{6}\sum_{r=4}^{98} r(r-1)\ (r-2)\,(r-3)}$$

$$\frac{= \frac{99}{6.5}\ \ \cdot(99.98.97.96.95) - \frac{1}{6.5}\displaystyle\sum_{r=4}^{98} r\big((r+1)\ r\ \ (r-1)(r-2)(r-3)\big) \quad - r(r-1)(r-2)(r-3)}{\frac{99}{6}\displaystyle\sum_{r=4}^{98}(r-1)(r-2)(r-3) - \frac{1}{6}\sum_{r=4}^{98} r(r-1)(r-2)(r-3)} \approx 66$$

18.(48)



For a Hexagone we can observe that there are 9 diagonals $(6_{c_2} - 6) = 9$ we can arrange in such a way that from A_1 (can take any point) we can draw 3 diagonal (6 – 3) & can take one diagonal (either A_2A_4 or A_4A_6) which can cut at only one diagonal. This will give us max 4 diagonal which satisfy the given situation. Similarly for so sided polygon.

Total diagonals for the above satisfying contains = (50 – 3) + 1 = 48

19.(92) $S(n) = 10^{p-1}x_1 + 10^{p-2}x_2 + ... + x_p$

$P(n) = x_1 + ... + x_p$

As $P(n)$ is free from square so $P(n)$ should be having only prime number as its digit and remain of multiple and $S(n)$ divides $P(n)$ so $S(n)$ should also be prime number (2, 3, 5, 7).

$S(n) \le 105$ (Proper divisors)

$S(n) = 7 \to P(n)_{\max} = 2 \times 3 \times 5 \times 7 \times k = 210$

$S(n) = 2+3+5+7+k\,(for\ k(1)) = 105$

$17 + k = 105 \quad \Rightarrow \quad k = 88$

Maximum possible digits = 92

20.(43) $A = \{x_1, ...x_n\}$

$B = \{y_1 y_2, ...y_n\}$

$x_n \times m = 12 \Rightarrow m = 1, x_n = 12 \to 11_{90}$

$y_m \times n = 11$

$n = 11, y_m = 1 \quad B = \{1\}$

$n = 1, y_m = 11$

$B = \{y_1, y_2, ...11\}$

$A = \{12\}$

$12 \times m = 12 \qquad m = 1 \to B\{11\} \to {}^{10}C_0$

$A = \{6\}$

$6 \times m = 12 \qquad m = 2 \to B = \{y_1, 11\} \to {}^{10}C_1$

$A = \{4\}$

$4 \times m = 12 \qquad m = 3 \to B = \{y_1, y_2, 11\} \to {}^{10}C_2$

$A = \{3\}$

$3 \times m = 12 \qquad m = 4 \to B\{y_1, y_2, y_3, 11\} \to {}^{10}C_3$

$2 \times m = 12$

$m = 6 \to B = \{y_1, y_2, y_3, y_4, 11\} \to {}^{10}C_5$

Total ordered pair

$= {}^{11}C_{10} + {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_5$

$= 22 + 45 + 120 + 252 = 429$

$a + b = 4 + 29 = 33$

21.(15) $\sum_{i=1}^{n}(i+f(i))=2023$

$\sum_{i=1}^{n} f(i)$ is least so $\sum_{i=1}^{n} i$ is maximum ≤ 2023

$$\frac{n(+1)}{2} \leq 2023$$

$\boxed{n=62}$ as minimum value of function can take is 1.

Number of functions such that

$n=63 \quad f(1)+f(2)+....+(63)=7$ not possible

$n=63 \quad f(1)+f(2)+....+(62)=70$

$f(1)+f(2)+....+(82)=70-62=8$

$f(1)+f(2)+.....+(62)=8$

$8 \to 8$

$8 \to (7,1)$

$8 \to 6,2$ or $(6,1,1)$

$8 \to 5,3$ or $(5, 2, 1)$ or $(5, 1, 1, 1)$

$8 \to (4, 4)$

Direction in 2 parts

(7, 1), (6, 2), (5, 2), (4, 4) in three parts

(6, 1, 1), (5, 2, 1), (1, 3, 4), (2, 2, 4), (2, 3, 3) in four parts

(5, 1, 1, 1), (4, 2, 1, 1), (3, 3, 1, 1), (2, 2, 2, 2) in five parts

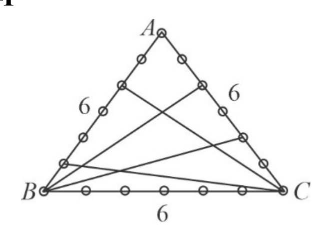
(4, 1, 1, 1, 1), (3, 2, 1, 1, 1), (2, 2, 2, 1, 1) in six parts

(3, 1, 1, 1, 1, 1), (2, 2, 1, 1, 1, 1) in seven parts

2, 1, 1, 1, 1, 1, 1 in eight parts 1, 1, 1, 1, 1, 1, 1, 1

Total functions = 21

22. **Case -I**

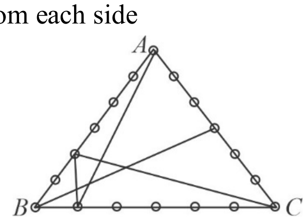


4 pegs should from 15 interior pegs

2 from each side ${}^{3}C_2 \times {}^{5}C_2 \times {}^{5}C_2 = 10 \times 10 \times 3 = 300$ sum of requires of digits = 9

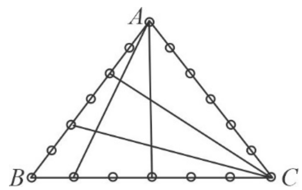
Case -II

One from each side



Only length page on will be formed so not possible.

22.



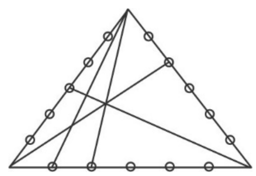
Case-I

2 pegs on 2 sides, zero on third side ${}^3C_2 \times {}^5C_2 \times {}^5C_2 = 300$

Case-II

2 pegs on one side and 1 peg each on remaining 2 sides

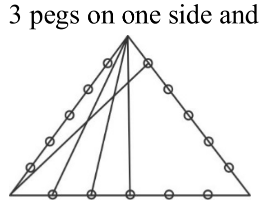
3 lines must be con current in this case



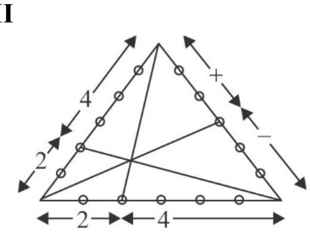
$\frac{1}{5}$

Case-III

3 pegs on one side and one on either of 2 sides hot possible



Case-II



	1 : 1			
→	1 : 5	1 : 5		
	2 : 4	2 : 4	1 : 5	5 : 1
	3 : 3	3 : 3	2 : 5	5 : 2 using care’s theorem
	4 : 2	4 : 2		
	5 : 1	5 : 1		

4 such cases exert for each side ratio 1 : 1 for three points

So total cases $= (({}^3C_4 \times 4) + 1) \times 12$ for fourth peg

$= 13 \times 12 = 156$

So total cases = 300 + 156 = 456

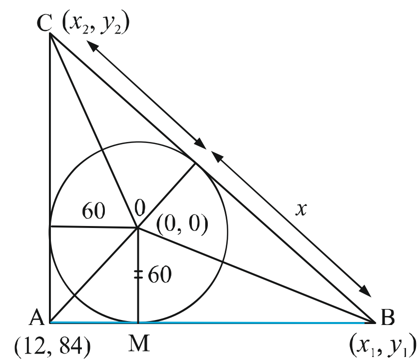
Sum of requires of digits = 16 + 25 + 36 = 77

23. $(x+y)^2 = (x+60)^2 + (y+60)^2$

$2xy = 120x + 120y$

$xy = 60(x \times y) \rightarrow \dfrac{x+y}{xy} = \dfrac{1}{60}$

$\dfrac{1}{x} \times \dfrac{1}{y} = \dfrac{1}{60}$



Centre is (0, 0)

$$0 = bx_1 + cx_2 + 12a$$

$$bx_1 + cx_1 + 12a = 0$$

$$by_1 + cy_2 + 84a = 0$$

$4y_2 - 3x_2 = 300$ …… (1)

$4x_1 + 3y_1 = 300$ …… (2)

Slope of $OA = \dfrac{84}{12} = 7$ (angle bisector)

Slope of AB & AC $\rightarrow$ M

$\tan 45° = \left|\dfrac{M-7}{1+7M}\right| = 1$

$M - 7 = 1 + 7M$ ‘+’

$6M = -8 \rightarrow M = -\dfrac{4}{3}$

$M - 7 = -1 - 7M;\quad 8M = 6 \quad \Rightarrow \quad M = \dfrac{3}{4}$

Equation of AC

$y - 84 = \dfrac{3}{4}(x-12)$

$4y - 336 = 3x - 36$

$4y - 3x = 300$

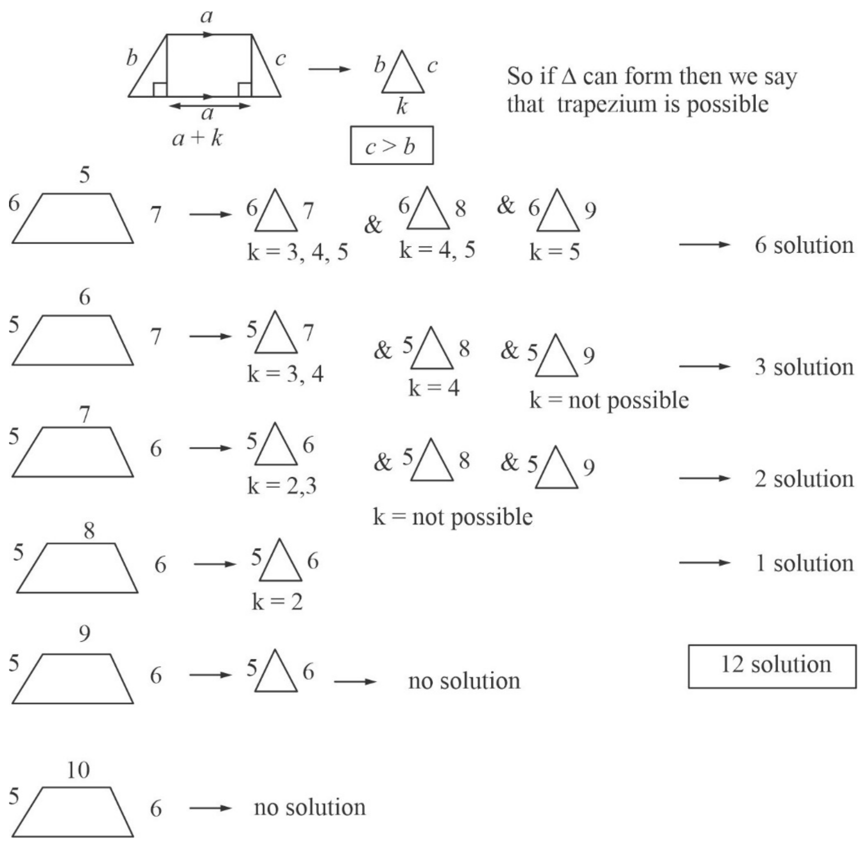
Equation of AB

$y - 84 = -\dfrac{4}{3}(x-12)$

$3y - 252 = -4x + 48$

$4x + 3y = 300$

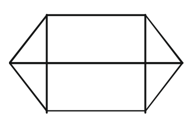
24.(12)



25. $D_n \geq 2000$

Number of such diagonal

$\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)$



$\frac{n}{2}-1 \rightarrow$ even

n must be of the form $4q+2$

$\frac{n}{2}=2q+1$

$\frac{n}{2}-1=2q$

$2q(2q+1)>1000$

$q(2q+1)>500$

$q=15$

$n_{\min}=62$

26.(100)

The Ben notes that can be used are

$$\begin{array}{l|l|l} 1-x_1 & 8-x_4 & 64-x_7 \\ 2-x_2 & 16-x_5 & \\ 4-x_3 & 32-x_6 & \end{array}$$

Now $x_1+2x_2+4x_3+8x_4+16x_5+32x_6+64x_7=100$

$0\le x_i \le 2i \qquad i=1,....., 6,\ x_7=0,1$

$x_i=0 \quad \text{or} \quad x_1=2$

If $x_1=0;\ x_2+2x_3+4x_4+8x_5+16x_6+32x_7=100$

27.(91) Total number $={}^{20}C_4=4845$

But the given coordinating of $S=4411$

Now for things to be balanced

$a+c=b+d$

$\Rightarrow\ c-d=b-a$

Say that $b-a=1$, for $a=1$, 17 choices

$a=2$, 16 choices

$\vdots$

$a=17$, 1 choice

$=1+2+3+...+16+17=17.9=153$

For $b-a=2$, $a=1$, 15 choices

$a=2$, 14 choices

$a=3$, 13 choices

$\vdots$

$a=17$, 1 choices

$=15\times\dfrac{16}{2}=120$

For $b-q=3$, $a=1 \quad 13+12+...+13\times\dfrac{14}{2}=91$

Similarly $11+10+....+1=11\times\dfrac{12}{2}=66$

And $9+8+....+1=9\times\dfrac{10}{2}=45$

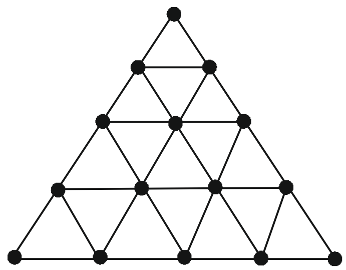
And $7+6+5+....+1=7\times\dfrac{8}{2}=28$

$5+4+....+1=5\times\dfrac{6}{2}=15$

$3+2+1+....+1=3\times\dfrac{4}{2}=6$

$1=1$

28.



Number of vertices $=1+2+3+..+n+(n+1)=\dfrac{(n+1)(n+2)}{2}$

Necessary condition is $(n+1)(n+2)=0$ mod (6)

Let's find the sufficient condition.

Trivially possible for $n=1$, similarly it works $n=2$

Claim $n=1,\ 2\ \text{mod}(4)$ is the sufficient condition

(Hint: 3 and 4 fail)

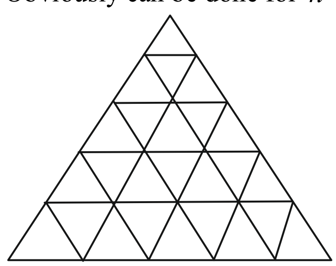
Proof: Via induction:

We can do it for $m=k$, then we can do it for $n=k+4$

$T\rightarrow$ A set of triangle for all vertices V, the number of triangles in s for which v should be odd.

Suppose if we can do it for $n=k$

Obviously can be done for $n=1$, now consider $n=5$



Colour every alternate triangle with Black and White and flip them.

So number of such values of $n=50$

29.(95) $99=11\cdot3\cdot3\cdot\underbrace{1\cdot1\cdot....\cdot1}_{82\text{ times}}=11+3+3+\underbrace{1+1+...+1}_{82\text{ times}}=33\cdot3\cdot\underbrace{1\cdot1\cdot....\cdot1}_{63\text{ times}}=33\cdot3+\underbrace{1+....+1}_{63\text{ times}}$

$98=49\cdot2\cdot\underbrace{1\cdot1....1}_{47\text{ times}}=49+2+\underbrace{1+1+....+1}_{47\text{ times}}=14\cdot7\cdot\underbrace{1\cdot....\cdot1}_{77\text{ times}}=14+7+\underbrace{1+1+....+1}_{77\text{ times}}$

$97=$ Prime number

$96=48\cdot2\cdot1\cdot1=16\cdot6\cdot1\cdot1$

$95=19\cdot5\cdot\underbrace{1\cdot1\cdot1\cdot....\cdot1}_{71\text{ times}}=19+5+\underbrace{1+1+....+1}_{71\text{ times}}$

That's the only way since 95 has only two prime divisors.

30.(18) Notice only perfect squares have odd number of divisors.

$$r=(2^2-1^2)+(4^2-3^2)+(6^2-5^2)+(8^2-7^2)+(10^2-9^2)+(12^2-11^2)+....+(44^2-43^2)$$

$$=1+2+3+4+....+43+44=(44)\frac{(45)}{2}=1980$$

Sum of digits $=1+9+8+0=18$