VMC Vidyamandir Classes SINCE 1986
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Indian Olympiad Qualifier in Mathematics (IOQM - 2023)

Note :

1. $\mathbb{N}$ denotes the set of all natural numbers, 1, 2, 3, ...

2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. For example, $\sqrt{4} = +2$.

3. Unless otherwise specified, all numbers are written in base 10.

Questions

1. Let n be a positive integer such that $1 \le n \le 1000$. Let M_n be the number of integers in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, ..., \sqrt{4n+1000}\}$.
Let $a = \max\{M_n : 1 \le n \le 1000\}$, *and* $b = \min\{M_n : 1 \le n \le 1000\}$. Find $a - b$.

2. Find the number of elements in the set $\{(a,b) \in \mathbb{N} : 2 \le a, b \le 2023, \log_a(b) + 6\log_b(a) = 5\}$

3. Let α and β be positive integers such that $\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$.
Find the smallest possible value of β.

4. Let x, y be positive integers such that $x^4 = (x-1)(y^3 - 23) - 1$.
Find the maximum possible value of $x + y$.

5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ.

6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

8. Given a 2 × 2 tile and seven dominoes (2 × 1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2 × 7 rectangle using some of these tiles.

9. Find the number of triples (a, b, c) of positive integers such that

(a) ab is a prime;

(b) bc is a product of two primes;

(c) abc is not divisible by square of any prime and

(d) $abc \le 30$.

10. The sequence $\langle a_n \rangle_{n \ge 0}$ is defined by $a_0 = 1, a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \ge 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

13. The ex-radii of a triangle are $10\frac{1}{2}$, 12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p + q + r}$.

14. Let ABC be a triangle in the xy plane, where B is at the origin (0, 0). Let BC be produced to D such that $BC : CD = 1:1$, CA be produced to E such that $CA : AE = 1:2$ and AB be produced to F such that $AB : BF = 1:3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

15. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN, and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m + n$.

16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

17. Consider the set $S = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$ where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.

18. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be *minimally friendly* if any diagonal $d \in F$ intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set $A = \{n \in \mathbb{N} : P(n)$ is non-zero, square free and $S(n)$ is a proper divisor of $P(n)\}$. Find the maximum possible number of digits of the number in A.

20. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12; \text{ and } |A| \times (B) = 11$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

21. For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways, we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point (12, 84). Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin (0, 0).

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set (5, 6, 7, 8, 9, 10).

25. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations $1, 2, 2^2, 2^3, \ldots$ Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 for two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27. A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a+c=b+d$. Let S be any set of quadruples (a,b,c,d) where $1 \le a < b < d < c \le 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in S.

28. One each side of an equilateral triangle with side length n units, where n is an integer $1 \le n \le 100$, consider $n-1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the side of the triangle, obtained a net of equilateral triangles of side length one unit. One each of the vertices of these small triangles, place a coin head up. Two coins are said to be adjacent if the distance between them 1 unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + ... + a_k = a_1 \cdot a_2 \cdots a_k$ for some positive integers $a_1, a_2, ..., a_k$, where $k > 1$ and $a_1 \ge a_2 \ge \cdots \ge a_k$. (For example 6 is beautiful since $6 = 3 \cdot 2 \cdot 1 = 3+2+1$, and this is unique. But 8 is not beautiful since $8 = 4+2+1+1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2+2+2+1+1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, so uniqueness is lost.) Find the largest beautiful number less than 100.

30. Let $d(m)$ denotes the number of positive integer divisors of a positive integer m. If r is the number of integers $n \le 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.