RMO	2023	Solutions

1. Since

$a^2 + b^2 + c^2 = d^2$

For any integer

$x,\ x^2 = 0 \text{ mod } (4)$

Or $x^2 = 1 \text{ mod } (4)$

So, $a^2 + b^2 + c^2 = 0 \text{ or } 1$ or 2

Or 3 (mod 4)

But $d^2 = 0 \text{ or } 1 (\text{mod } 4)$

For a solution to exist

$a^2 + b^2 + c^2 = 0 \text{ mod } (4)$

Or 1 mod (4)

In either case of the case at least two numbers are even

Now for any integer x

$x^2 = 0 \text{ mod } (3)$ or

$x^2 = 1 \text{ mod } (3)$

$\Rightarrow a^2 + b^2 + c^2 = 0 \text{ mod}(3)$ Or 1 mod (3)

$d^2 = 0 \text{ mod } (3)$

Or 1 mod (3)

At least one of the numbers a, b, c, d is divisible by 3.

So, $a \cdot b \cdot c \cdot d$ is divisible by 12

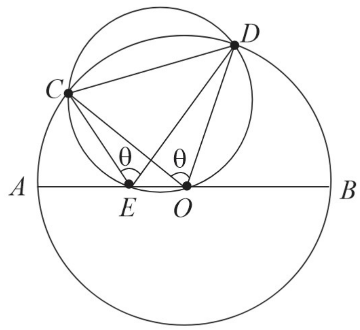
And $1^2 + 2^2 + 2^2 = 3^2$

$(1 \cdot 2 \cdot 2 \cdot 3)$ is divisible by 12.

So, the answer is 12.

2. Let O be the centre of the circle curve CD is of fixed length so $\angle COD$ must be fixed.

Claim: the point P where the circumference of triangle COD intersects the diameter AB is the point where $\angle AEC = \angle BED = \frac{\pi}{2} - \frac{\theta}{2}$ and the circumcircle of CED passes through O, the fixed point.



3. $(S(m))^2 = n$

And $(S(n))^2 = m$

Now, m and n must be perfect squares.

Obviously single digit numbers do Not work.

Say $n = ab$ and $m = cd$

$1000\,ab = (c+d)^2 \; and 100\,c + d = (a+b)^2$

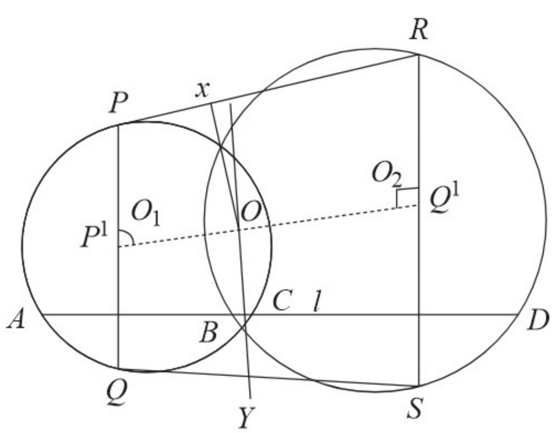
Which is not possible.

Consider three digit numbers which are perfect squares 169 and 256 fit the bill.

No number greater than 4 digit number satisfies this because $n > (S(m))^2$

4. O is the mid-point of O_1O_2 using opposite of ΔPT

XOY are collinear



5. $n > k > 1$ be positive integer $\sum_{i=1}^{n} \sqrt{\dfrac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n$

$a_{ij} = 0.$

$$\frac{\overbrace{a_i^k + a_i^k + \ldots + a_i^k}^{}}{\underset{k}{(k-1)\,times}} + 1 \geq a_i^{k-1} \quad \Rightarrow \quad (k-1)\, a_i^k + 1 \geq k a_i^{k-1}$$

Or $\quad \dfrac{ka_i^k}{(k-1)a_i^k + 1} \leq a_i$

$$\Rightarrow \quad \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \leq \sqrt{a_i} \quad \Rightarrow \quad \sum_{i=1}^{n} a_i = n \leq \sum_{i=1}^{n} a_i$$

$$n^2 \leq \left(\sum \sqrt{a_i}\right)^2$$

Using Cauchy-schwarz

$$(1^2 + 1^2 + 1^2 + \ldots + 1^2)\,(a_1 + a_2 + \ldots + a_n) \geq (\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n})^2$$

$$n \cdot n \geq \left(\sum_{i=1}^{n} a_i\right)^2 \qquad \Rightarrow \qquad \sum_{i=1}^{n} a_i^2 = n^2$$

Equality holds $\Rightarrow \quad a_1 = a_2 = \ldots. = a_n = 1$

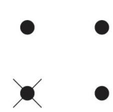
Hence proved

6. Motivation:

$(4 \times 4) - 7 = (3 \times 3)$

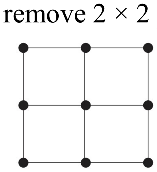
So, it's as good as saying you are removing (3 × 3) lattice points from a (4 × 4) grid and that still guarantees that there is an isosceles right triangle.

Case – I: Consider a 2 × 2 grid and let's remove 1 × 1 point.



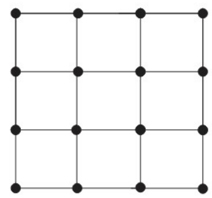
There is obviously an isosceles right triangle.

Case – II: Consider a 3 × 3 grid and lets remove 2 × 2 points.



A 3×3 grid can be decomposed into 4 2×2 grids. From 4 points which are to be removed if all lies in one grid each, then it's the previous case and we have an isosceles right triangle. Else we can take more points from one grid and that gives us a 2×2 isosceles right triangle.

Consider a 4×4 grid



It contains 4 3×3 grid. Now we have to take away 9 points.

Obviously at least one grid will have 3 points removed leaving others with 2 or forever points to be removed. Hence the result holds.